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Projection on higher Landau levels and non-commutative geometry

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Abstract
The projection of a two-dimensional planar system on the higher Landau levels of an external magnetic field is formulated in the language of the non-commutative plane and leads to a new class of star products.

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1. Introduction

Consider a two-dimensional Hamiltonian of a particle in a scalar potential $V(z,\bar{z})$ coupled to a magnetic field B, in the symmetric gauge ($e = \hbar = m = 1$)

$$H(z) = -2\partial\bar{\partial} + \omega_c(\bar{z}\bar{\partial} - z\partial) + \tfrac{1}{2}\omega_c^2 z\bar{z} + V(z,\bar{z}). \tag{1}$$

We assume without any loss of generality that $B \geqslant 0$ and denote by $\omega_c = +B/2$ half the cyclotron frequency. The model (1), with the potential V random due to impurities, is central to the present understanding of the integer quantum Hall effect [1].

It is well known that by projecting (1) on the lowest Landau level (LLL) spanned by the states

$$\psi(z) = f(z)\,\mathrm{e}^{-\frac{\omega_c}{2}z\bar{z}} \tag{2}$$

where $f(z)$ is analytic, one obtains an eigenvalue equation

$$:V\left(z, \frac{1}{\omega_c}\partial_z\right):\, f(z) = (E - \omega_c) f(z). \tag{3}$$

The normal ordering : : means that the differential operator $\frac{1}{\omega_c}\partial$ is put on the left of z. Equation (3) is a reformulation of the Peierls substitution [2] (which does not specify the correct ordering in general) and as such has been derived in [3] (which specifies the correct ordering).

Clearly, the usual two-dimensional plane has been traded for a non-commutative space

$$\left[\frac{1}{\omega_c}\partial_z, z\right] = \frac{1}{\omega_c}. \tag{4}$$

In view of the above commutation relation, the non-commutative space can be interpreted as the phase space corresponding to a one-dimensional space. However this interpretation may not be entirely satisfactory because z is a complex coordinate. There is another natural interpretation in terms of a non-commutative space which is a two-dimensional plane with non-commuting 'real' coordinates (X, Y). This point of view has been recently used in the context of the quantum Hall effect [4]. The algebra of operators depending on the two non-commutative coordinates (X, Y) is equivalent [5] to a deformation of the classical algebra of functions on the usual commutative plane with coordinates (x, y). This deformation is defined through a non-commutative star product. We will review this construction below in a self-contained way, giving, as a by-product, a more systematic and more simple derivation of (3).

The main point of this paper is to show that non-commutative geometry is by no means specific to the LLL projection but can be obtained as well by projecting the two-dimensional system on any given higher Landau level. As an illustration the case of the first Landau level (1LL) will be analysed in detail. We will generalize the Peierls substitution to the 1LL and reformulate it in a non-commutative plane language. We will find that although the canonical commutation relation between the non-commutative coordinates will be the same as for the LLL, a new non-commutative star product will appear to be naturally connected with the 1LL. We will then give the general expression for the star product associated with any given Landau level, thereby defining a new class of star products [6].

It might be objected that projecting a system on a given higher Landau level is counter-intuitive: usually the LLL projection is regarded as physically justified when the cyclotron gap $\hbar\omega_c$ is sufficiently large. Thus the LLL projection is associated with a strong magnetic field compared to the temperature and/or to the potential so that the excited states above the LLL can be ignored. For an electron gas the filling of the Fermi–Landau sea up to the nth Landau level requires us to consider a projection on all n levels. In many instances one can neglect the mixing between different levels so that each level is treated separately. In any case, restricting the two-dimensional Hilbert space to a given Landau level subspace is a well-defined mathematical procedure, and will be considered as such in what follows.

Before starting let us recall that the Landau spectrum is made of degenerate Landau levels $(2n+1)\omega_c$, $n \geqslant 0$ with the nth Landau level eigenstates labelled by the radial/orbital quantum numbers $n, l \geqslant 0$ (analytic) and $n+l, -n \leqslant l < 0$ (anti-analytic). There are, in a given Landau level, an infinite number of analytic eigenstates

$$\langle z, \bar{z}|n, l\rangle = z^l L_n^l(\omega_c z\bar{z})\,\mathrm{e}^{-\frac{1}{2}\omega_c z\bar{z}} \qquad l \geqslant 0 \tag{5}$$

and a finite number of anti-analytic eigenstates

$$\langle z, \bar{z}|n+l, l\rangle = \bar{z}^{-l} L_{n+l}^{-l}(\omega_c z\bar{z})\,\mathrm{e}^{-\frac{1}{2}\omega_c z\bar{z}} \qquad -n \leqslant l < 0. \tag{6}$$

2. Projection on the LLL

In the LLL, $n = 0$, $l \geqslant 0$, the eigenstates are analytic

$$\langle z, \bar{z}|0, l\rangle = \left(\frac{\omega_t^{\ell+1}}{\pi\ell!}\right)^{\frac{1}{2}} z^\ell\,\mathrm{e}^{-\frac{1}{2}\omega_t z\bar{z}} \qquad \ell \geqslant 0. \tag{7}$$

Consider the projector on the LLL Hilbert space $P_o = \sum_{l=0}^{\infty} |0, l\rangle\langle 0, l|$

$$\langle z, \bar{z}|P_o|z', \bar{z}'\rangle = \frac{\omega_c}{\pi}\, \mathrm{e}^{-\frac{1}{2}\omega_c(z\bar{z}+z'\bar{z}'-2z\bar{z}')}. \tag{8}$$

A state of the LLL $|\psi\rangle = \sum_{l=0}^{\infty} a_l|0, l\rangle$, is analytic up to the Landau–Gaussian factor

$$\langle z|\psi\rangle = f(z)\, \mathrm{e}^{-\frac{1}{2}\omega_c z\bar{z}} \tag{9}$$

with

$$f(z) = \sum_{l=0}^{\infty} a_l z^l. \tag{10}$$

One can check that $|\psi\rangle$ satisfies $P_o|\psi\rangle = |\psi\rangle$.

Then projecting the Hamiltonian (1) on the LLL

$$\langle z, \bar{z}|P_o H P_o|\psi\rangle = \langle z, \bar{z}|P_o H|\psi\rangle = \int \mathrm{d}z'\, \mathrm{d}\bar{z}' \frac{\omega_c}{\pi}\, \mathrm{e}^{-\frac{1}{2}\omega_c(z\bar{z}+z'\bar{z}'-2z\bar{z}')} H(z') f(z')\, \mathrm{e}^{-\frac{1}{2}\omega_c z'\bar{z}'}. \tag{11}$$

Using the Bargman identity

$$\frac{\omega_c}{\pi} \int \mathrm{d}z'\, \mathrm{d}\bar{z}'\, \mathrm{e}^{-\omega_c(z'\bar{z}'-z\bar{z}')} h(z') = h(z) \tag{12}$$

it is immediate to see that the eigenvalue equation

$$\langle z, \bar{z}|P_o H P_o|\psi\rangle = E\langle z, \bar{z}|\psi\rangle \tag{13}$$

or

$$\int \mathrm{d}z'\, \mathrm{d}\bar{z}' \frac{\omega_c}{\pi}\, \mathrm{e}^{-\omega_c(z'\bar{z}'-z\bar{z}')}(\omega_c + V(z', \bar{z}')) f(z') = E f(z) \tag{14}$$

can be transformed into a differential equation which is precisely the Peierls substitution equation (3).

3. Projection on the 1LL

Consider now the projection on the 1LL Hilbert space $P_1 = \sum_{l=0}^{\infty} |1, l\rangle\langle 1, l| + |0, -1\rangle\langle 0, -1|$

$$\langle z, \bar{z}|P_1|z', \bar{z}'\rangle = \langle z, \bar{z}|P_0|z', \bar{z}'\rangle[1 - \omega_c(z' - z)(\bar{z}' - \bar{z})]. \tag{15}$$

Using $L_1^l(\omega_c z\bar{z}) = l + 1 - \omega_c z\bar{z}$ one obtains that a state of the 1LL $|\psi\rangle = \sum_{l=0}^{\infty} a_l|1, l\rangle + a_{-1}|0, -1\rangle$ is of the form

$$\langle z, \bar{z}|\psi\rangle = (f(z) + \bar{z}g(z))\, \mathrm{e}^{-\frac{1}{2}\omega_c z\bar{z}} \tag{16}$$

with $f(z)$ and $g(z)$ analytic,

$$f(z) = -\frac{1}{\omega_c}\partial g(z) \tag{17}$$

and

$$g(z) = -\omega_c \sum_{l=0}^{\infty} a_l z^{l+1} + a_{-1}. \tag{18}$$

Another way to recover this result is to impose equivalently that $P_1|\psi\rangle = |\psi\rangle$, i.e.,

$$\int \mathrm{d}z'\, \mathrm{d}\bar{z}' \langle z, \bar{z}|P_1|z', \bar{z}'\rangle\langle z', \bar{z}'|\psi\rangle = \langle z, \bar{z}|\psi\rangle. \tag{19}$$

One first infers that necessarily $\langle z, \bar{z}|\psi\rangle = (f(z) + \bar{z}g(z))\,\mathrm{e}^{-\frac{1}{2}\omega_c z\bar{z}}$. Equation (19) then becomes

$$\int \mathrm{d}z'\,\mathrm{d}\bar{z}'\frac{\omega_c}{\pi}\,\mathrm{e}^{-\omega_c(z'\bar{z}'-z\bar{z}')}[1-\omega_c(z'-z)(\bar{z}'-\bar{z})](f(z')+\bar{z}'g(z')) = f(z)+\bar{z}g(z) \tag{20}$$

and finally

$$-\frac{1}{\omega_c}\partial g(z) + \bar{z}g(z) = f(z) + \bar{z}g(z). \tag{21}$$

The relation (17) directly follows.

Now we project the Hamiltonian (1) on the 1LL,

$$\langle z, \bar{z}|P_1 H P_1|\psi\rangle = \int \mathrm{d}z'\,\mathrm{d}\bar{z}'\langle z, \bar{z}|P_1|z', \bar{z}'\rangle\langle z', \bar{z}'|H|\psi\rangle. \tag{22}$$

The eigenvalue equation

$$\langle z, \bar{z}|P_1 H P_1|\psi\rangle = E\langle z, \bar{z}|\psi\rangle \tag{23}$$

becomes

$$\int \mathrm{d}z'\,\mathrm{d}\bar{z}'\frac{\omega_c}{\pi}\,\mathrm{e}^{-\omega_c(z'\bar{z}'-z\bar{z}')}[1-\omega_c(z'-z)(\bar{z}'-\bar{z})]V(z',\bar{z}')(f(z')+\bar{z}'g(z'))$$
$$= (E-3\omega_c)(f(z)+\bar{z}g(z)). \tag{24}$$

Using (12) again one transforms (24) into the differential equation

$$-\frac{1}{\omega_c}\partial\left(: V + \frac{1}{\omega_c}\partial\bar{\partial}V : g(z)\right) + \bar{z} : V + \frac{1}{\omega_c}\partial\bar{\partial}V : g(z) = (E-3\omega_c)(f(z)+\bar{z}g(z)) \tag{25}$$

where it is again understood that $\bar{z}$ is replaced by $\frac{1}{\omega_c}\partial$ both in V and in $\partial\bar{\partial}V$, and then the normal ordering is taken.

As an example, consider $V(z,\bar{z}) = \omega_c^2 z\bar{z}/2$. We obtain from (25)

$$\begin{aligned}&\frac{\omega_c}{2}(3+z\partial)g = (E-3\omega_c)g(z)\\ &\frac{\omega_c}{2}(4+z\partial)f = (E-3\omega_c)f(z).\end{aligned} \tag{26}$$

It is easy to see that the pair of equations lead to the spectrum $E - 3\omega_c = \omega_c(3+l)/2, l \geqslant 0$. For a general $V(z,\bar{z})$ this conclusion still holds true: the 1LL projection induces an eigenvalue equation

$$:V\left(z, \frac{1}{\omega_c}\partial\right) + \frac{1}{\omega_c}\bar{\partial}\partial V\left(z, \frac{1}{\omega_c}\partial\right) : g(z) = (E-3\omega_c)g(z) \tag{27}$$

where $g(z)$ is analytic. The equation for $f(z)$ is obtained by differentiating (27) with respect to z. Equation (27) is, in the 1LL, the analogue of the LLL Peierls substitution equation (3).

4. Non-commutative plane and $\star$ products

LLL quantum mechanics can be equivalently reformulated in a non-commutative geometry setting. It is tempting to generalize this construction to higher Landau levels, which results in a class of non-commutative $\star_n$ products associated with the higher Landau level projection. Let us first start by recalling the LLL non-commutative formulation.

4.1. Lowest Landau level

Consider the non-commutative guiding centre coordinates

$$X = \frac{1}{2}\left(z + \frac{1}{\omega_c}\partial\right) \qquad Y = \frac{1}{2\mathrm{i}}\left(z - \frac{1}{\omega_c}\partial\right). \tag{28}$$

They satisfy the commutation relation

$$[X, Y] = \frac{1}{2\mathrm{i}\omega_c}. \tag{29}$$

We have seen in (3) that the LLL operator associated with the classical potential $V(x, y)$ is

$$:V\left(z, \frac{1}{\omega_c}\partial z\right): . \tag{30}$$

Introducing the Fourier transform of the classical potential

$$V(x, y) = \int \mathrm{d}k\,\mathrm{d}l\,\mathrm{e}^{\mathrm{i}(kx+ly)}\tilde{V}(k, l) = \int \mathrm{d}k\,\mathrm{d}l\,\mathrm{e}^{\mathrm{i}(\frac{k}{2}(z+\bar{z})+\frac{l}{2\mathrm{i}}(z-\bar{z}))}\tilde{V}(k, l) \tag{31}$$

and using (28) the operator (30) can be written as

$$\hat{V}^{(0)}(X, Y) = \int \mathrm{d}k\,\mathrm{d}l : \mathrm{e}^{\mathrm{i}kX+\mathrm{i}lY} : \tilde{V}(k, l) \tag{32}$$

inducing a mapping between a classical potential and an operator. Note that (32) is different from the Weyl ordering which is often used in the field theory context [5] and corresponds to dropping the normal order double colons in (32). Note also that for further use, thanks to the identity

$$:\mathrm{e}^{\mathrm{i}kX+\mathrm{i}lY} := \mathrm{e}^{-\frac{1}{8\omega_c}(k^2+l^2)}\mathrm{e}^{\mathrm{i}kX+\mathrm{i}lY} \tag{33}$$

one has

$$\hat{V}^{(0)}(X, Y) = \int \mathrm{d}k\,\mathrm{d}l\,\mathrm{e}^{-\frac{1}{8\omega_c}(k^2+l^2)}\,\mathrm{e}^{\mathrm{i}kX+\mathrm{i}lY}\,\tilde{V}(k, l). \tag{34}$$

Equation (32) induces a non-commutative product $f \star_0 g$ between any two classical functions $f(x, y)$ and $g(x, y)$ such that

$$\hat{f}^{(0)}(X, Y)\hat{g}^{(0)}(X, Y) = (f \mathbin{\widehat{\star_0}} g)^{(0)}(X, Y). \tag{35}$$

A straightforward computation leads to

$$(f \star_0 g)(x, y) = \mathrm{e}^{-\frac{1}{4\omega_c}(\partial'_x+\mathrm{i}\partial'_y)(\partial_x-\mathrm{i}\partial_y)} f(x, y)g(x', y')|_{x=x', y=y'} \tag{36}$$

and

$$(f \mathbin{\widetilde{\star_0}} g)(k, l) = \int \mathrm{d}k'\,\mathrm{d}l'\,\tilde{f}(k-k', l-l')\tilde{g}(k', l')\,\mathrm{e}^{\frac{\mathrm{i}}{4\omega_t}(kl'-lk')}\,\mathrm{e}^{-\frac{1}{4\omega_c}(k'^2+l'^2-kk'-ll')}. \tag{37}$$

The Fourier transform (37) of $f \star_0 g$ is a 'deformation' of the usual convolution product of the Fourier transforms of f and g. The canonical commutation relation may be expressed as

$$x \star_0 y - y \star_0 x = \frac{1}{2\mathrm{i}\omega_c}. \tag{38}$$

In terms of the non-commutative coordinates (X, Y) the eigenvalue equation (3) acting on $f(z)$ can be rewritten as

$$\hat{V}^{(0)}(X, Y)f(X + \mathrm{i}Y) = (E - \omega_c)f(X + \mathrm{i}Y). \tag{39}$$

Looking at $f(X + \mathrm{i}Y) \equiv \hat{f}^{(0)}$ as an operator and using (34) one finds that (39) is equivalent to

$$(V \mathbin{\widehat{\star_0}} f)^{(0)}(X, Y) = (E - \omega_c)f(X + \mathrm{i}Y). \tag{40}$$

In other words the eigenvalues of (3) and of the operator equation (40) are identical. Note finally that the analyticity of f implies $V \star_0 f = Vf$, and therefore $(\widehat{Vf})^{(0)}(X, Y) = (E - \omega_c)f(X + \mathrm{i}Y)$.[3]

[3] Of course it does not imply that $Vf(x + \mathrm{i}y) = (E - \omega_c)f(x + \mathrm{i}y)$.

4.2. First Landau level

As seen in (27), the 1LL operator associated with the potential $V(x, y)$ is

$$:V\left(z, \frac{1}{\omega_c}\partial\right) + \frac{1}{\omega_c}\bar{\partial}\partial V\left(z, \frac{1}{\omega_c}\partial\right): . \tag{41}$$

This can be re-expressed as

$$\hat{V}^{(1)}(X, Y) = \int \mathrm{d}k\,\mathrm{d}l : \mathrm{e}^{\mathrm{i}kX+\mathrm{i}lY} : \left(1 - \frac{1}{4\omega_c}(k^2 + l^2)\right)\tilde{V}(k, l). \tag{42}$$

Using the commutation relation (29) one can check that

$$(k^2 + l^2)\,\mathrm{e}^{\mathrm{i}kX+\mathrm{i}lY} = [X, [X, \mathrm{e}^{\mathrm{i}kX+\mathrm{i}lY}] + [Y, [Y, \mathrm{e}^{\mathrm{i}kX+\mathrm{i}lY}]. \tag{43}$$

Thus from (33) and (43) we find

$$\hat{V}^{(1)}(X, Y) = \hat{V}^{(0)}(X, Y) + \frac{1}{4\omega_c}\hat{\Delta}\hat{V}^{(0)}(X, Y) \tag{44}$$

where in (44)

$$\hat{\Delta} = [X, [\cdot, X] + [Y, [\cdot, Y] \tag{45}$$

and

$$(\widehat{\Delta V})^{(0)}(X, Y) = \hat{\Delta}\hat{V}^{(0)}(X, Y) \tag{46}$$

is understood. Equation (43) suggests that the operator (45) is the non-commutative version of the Laplacian [7].

The 1LL mapping (42) between a classical function and an operator induces a new star product $\star_1$ such that for any two classical functions $f(x, y)$ and $g(x, y)$

$$\hat{f}^{(1)}(X, Y)\hat{g}^{(1)}(X, Y) = (f \,\widehat{\star_1}\, g)^{(1)}(X, Y). \tag{47}$$

A differential expression for the $\star_1$ product can easily be found using (46). Indeed,

$$\begin{aligned} \hat{f}^{(1)}\hat{g}^{(1)} &= \left(\hat{f}^{(0)} + \frac{1}{4\omega_c}\hat{\Delta}\hat{f}^{(0)}\right)\left(\hat{g}^{(0)} + \frac{1}{4\omega_c}\hat{\Delta}\hat{g}^{(0)}\right) \\ &= \left(\hat{f}^{(0)} + \frac{1}{4\omega_c}\widehat{\Delta f}^{(0)}\right)\left(\hat{g}^{(0)} + \frac{1}{4\omega_c}\widehat{\Delta g}^{(0)}\right) \end{aligned} \tag{48}$$

so that

$$f \star_1 g = \left(f + \frac{1}{4\omega_c}\Delta f\right) \star_0 \left(g + \frac{1}{4\omega_c}\Delta g\right). \tag{49}$$

Thus

$$f \star_1 g = \mathrm{e}^{-\frac{1}{4\omega_c}(\partial_x - \mathrm{i}\partial_y)(\partial'_x + \mathrm{i}\partial'_y)}\left(1 + \frac{1}{4\omega_c}\Delta\right)\left(1 + \frac{1}{4\omega_c}\Delta'\right) f(x, y)g(x', y')|_{x=x', y=y'}. \tag{50}$$

Note that it follows from (50) that the canonical commutation relation $x \star_1 y - y \star_1 x = \frac{1}{2\mathrm{i}\omega_c}$ is identical to (38), i.e. it is the same as in the LLL.

In terms of the non-commutative coordinates X, Y the eigenvalue equation (27) becomes

$$\hat{V}^{(1)}(X, Y)g(X + \mathrm{i}Y) = (E - 3\omega_c)g(X + \mathrm{i}Y) \tag{51}$$

or equivalently

$$(V \,\widehat{\star_1}\, g)^{(1)}(X, Y) = (E - 3\omega_c)g(X + \mathrm{i}Y) \tag{52}$$

Note that since $g(z)$ is analytic $V \star_1 g = \left(V + \frac{1}{4\omega_c}\Delta V\right)g$. Therefore (52) is nothing but $\left(\widehat{\left(V + \frac{1}{4\omega_c}\Delta V\right)g}\right)^{(1)}(X, Y) = (E - 3\omega_c)g(X + \mathrm{i}Y)$.

4.3. Higher Landau level

Let us apply the same procedure as in the LLL and the 1LL to the nth Landau level (nLL). An eigenstate is of the form

$$\langle z, \bar{z}|\psi\rangle = \big(f_0(z) + \bar{z} f_1(z) + \cdots + \bar{z}^n f_n(z)\big)\, \mathrm{e}^{-\frac{\omega_c}{2} z\bar{z}} \tag{53}$$

with $f_n(z)$ analytic and

$$f_i(z) = \omega_c^{(i-n)}(-1)^{(n-i)} \frac{n!}{i!(n-i)!} \frac{\partial^{n-i}}{\partial z^{n-i}} f_n(z). \tag{54}$$

Projecting the Hamiltonian (1) on the nLL implies that $f_n(z)$ satisfies the eigenvalue equation

$$:\sum_{i=0}^{n} \frac{1}{\omega_c^i} \frac{n!}{i!^2(n-i)!} \frac{\partial^{2i}}{\partial z^i \partial \bar{z}^i} V: f_n(z) = (E - (2n+1)\omega_c) f_n(z) \tag{55}$$

which can be viewed as the generalization of the LLL Peierls substitution equation (3) to the nLL (details of the derivation of (53) and (55) can be found in [8]).

In a non-commutative plane formulation, the potential $V(x, y)$ is replaced by the nLL operator

$$\hat{V}^{(n)}(X, Y) = \int \mathrm{d}k\, \mathrm{d}l : \mathrm{e}^{\mathrm{i}kX + \mathrm{i}lY} : \tilde{V}(k, l) \sum_{i=0}^{n} \left(-\frac{1}{4\omega_c}\right)^i \frac{n!}{i!^2(n-i)!}(k^2 + l^2)^i \tag{56}$$

with also

$$\hat{V}^{(n)}(X, Y) = \sum_{i=0}^{n} \left(\frac{1}{4\omega_c}\right)^i \frac{n!}{i!^2(n-i)!} (\hat{\Delta})^i \hat{V}^{(0)}(X, Y). \tag{57}$$

This induces a $\star_n$ product between classical functions such that $\hat{f}^{(n)}\hat{g}^{(n)} = (f \mathbin{\widehat{\star_n}} g)^{(n)}$: namely one has

$$f \star_n g = \mathrm{e}^{-\frac{1}{4\omega_c}(\partial_x - \mathrm{i}\partial_y)(\partial'_x + \mathrm{i}\partial'_y)} D^{(n)}_{x,y} D^{(n)}_{x',y'} f(x, y) g(x', y')|_{x=x', y=y'} \tag{58}$$

where $D^{(n)}_{x,y} = \sum_{i=0}^{n} \left(\frac{1}{4\omega_c}\right)^i \frac{n!}{i!^2(n-i)!} (\Delta)^i$. Note that the canonical commutation relation associated with $\star_n$ again narrows down to (38), i.e. to the LLL situation.

4.4. Hilbert space

Equation (56) gives a natural specification of nLL operators $\hat{f}^{(n)}(X, Y)$ associated with classical functions $f(x, y)$. One can also construct a Hilbert space on which these operators act and which turns out to be equivalent to the Bargmann space of analytic functions with the scalar product $\langle f|g\rangle = (\omega_c/\pi) \int \mathrm{d}z\, \mathrm{d}\bar{z} e^{-\omega_c z\bar{z}} \bar{f} g$.

Let us define the creation–annihilation operators $A^\dagger = \omega_c^{1/2}(X + \mathrm{i}Y)$ and $A = \omega_c^{1/2}(X - \mathrm{i}Y)$ and the corresponding Hilbert space spanned by the vectors $A^{\dagger n}|0\rangle$ acting on the vacuum $|0\rangle$ with $A|0\rangle = 0$. Since $[A, A^\dagger] = 1$ we have

$$\langle 0|\, \mathrm{e}^{\mathrm{i}(kX + lY)}|0\rangle = \mathrm{e}^{-\frac{1}{2}\langle 0|(kX + lY)^2|0\rangle} = \mathrm{e}^{-\frac{1}{8\omega_c}(k^2 + l^2)}. \tag{59}$$

If we now consider in the Hilbert space two states $\hat{f}^{(n)}|0\rangle$ and $\hat{g}^{(n)}|0\rangle$, their scalar product becomes

$$\begin{aligned}\langle 0|\hat{f}^{(n)\dagger}\hat{g}^{(n)}|0\rangle &= \langle 0|(\bar{f} \mathbin{\widehat{\star_n}} g)^{(n)}|0\rangle \\ &= \int \mathrm{d}k\, \mathrm{d}l\, \mathrm{e}^{-\frac{1}{4\omega_c}(k^2 + l^2)} (\bar{f} \mathbin{\widetilde{\star_n}} g)(k, l) \sum_{i=0}^{n} \left(-\frac{1}{4\omega_c}\right)^i \frac{n!}{i!^2(n-i)!}(k^2 + l^2)^i \\ &= \frac{\omega_c}{\pi} \int \mathrm{d}x\, \mathrm{d}y\, \mathrm{e}^{-\omega_c(x^2 + y^2)} D^{(n)}_{x,y} (\bar{f} \star_n g)(x, y). \end{aligned} \tag{60}$$

The scalar product (60) is equivalent to the scalar product of the Bargmann space. Indeed for any classical function $f(x, y)$, one can always define a polynomial (analytic) function of $A^\dagger$ such that $\hat{f}^{(n)}|0\rangle = p_f(A^\dagger)|0\rangle$. The analyticity then implies that $p_f \star_n p_g = p_f p_g$ so that

$$\langle 0|\hat{f}^{(n)\dagger}\hat{g}^{(n)}|0\rangle = \frac{\omega_c}{\pi}\int \mathrm{d}x\,\mathrm{d}y\,\mathrm{e}^{-\omega_c(x^2+y^2)}\,\bar{p}_f(x-\mathrm{i}y)p_g(x+\mathrm{i}y). \tag{61}$$

Equations (40) and (52) can be viewed as *bona fide* eigenvalue equations on the Hilbert space spanned by $(A^\dagger)^n|0\rangle$,

$$\hat{V}^{(n)}(X, Y)f(X+\mathrm{i}Y)|0\rangle = (E-(2n+1))f(X+\mathrm{i}Y)|0\rangle. \tag{62}$$

Note finally that in (60) scalar products have been expressed in the operator language or equivalently in terms of star products, leading to nontrivial identities[4].

5. Conclusion

One has obtained the class of star products $\star_n$ which generalize to the nth Landau level the standard $\star_0$ product in the LLL. A non-commutative space can be thus defined each time the two-dimensional Hilbert space is projected on a given Landau level. Accordingly, a nLL Peierls subtsitution equation is obtained which generalizes the standard LLL Peierls substitution equation. In each nLL subspace the space is non-commutative with a canonical commutation relation $x \star_n y - y \star_n x = \frac{1}{2\mathrm{i}\omega_c}$. However, by considering altogether all the nLL projections one should recover the full Landau spectrum and therefore the commutative two-dimensional space.

References

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[4] The scalar product

$$\langle 0|\hat{f}^{(n)\dagger}\hat{f}^{(n)}|0\rangle = \frac{\omega_c}{\pi}\int \mathrm{d}x\,\mathrm{d}y\,\mathrm{e}^{-\omega_c(x^2+y^2)}D^{(n)}_{x,y}(\bar{f}\star_n f)(x, y) \tag{A}$$

must be non-negative for any $f(x, y)$ although

$$D^{(n)}_{x,y}(\bar{f}\star_n f)(x, y) = \sum_{j=0}^{\infty}\frac{(-1)^j}{j!}(4\omega_c)^{-j}D^{(n)}_{x,y}|(\partial_x+\mathrm{i}\partial_y)^j D^{(n)}_{x,y}f(x, y)|^2 \tag{B}$$

is an alternating sum. As an illustration consider in the LLL ($n = 0$) $f(x, y) = (x+\mathrm{i}y)^k(x-\mathrm{i}y)^l$. Then $\hat{f}^{(0)}(X, Y) = (X-\mathrm{i}Y)^l(X+\mathrm{i}Y)^k$ so that for $l > k$ we have $\hat{f}^{(0)}|0\rangle = 0$. This is satisfied if, evaluating the right hand side of (A), the combinatorial identity

$$\sum_{j=0}^{l}(-1)^j\frac{(j+k)(j+k-1)\ldots(j+1)}{(l-j)!j!} = 0 \qquad l > k \geqslant 0 \tag{C}$$

is verified. An independent check can be made using the binomial expansion of $\frac{\mathrm{d}^k}{\mathrm{d}x^k}(1-x)^l|_{x=1} = 0$ for $l > k \geqslant 0$.